

The action of pseudo-differential operators on functions harmonic outside a smooth hyper-surface

Louis Boutet de Monvel* & Yves Colin de Verdière†

September 25, 2012

The goal of this note is to describe the action of pseudo-differential operators on the space \mathcal{H} of L^2 functions which are harmonic outside a smooth closed hyper-surface Z of a compact Riemannian manifold without boundary (X, g) and whose traces from both sides of Z coincide. We will represent these L^2 harmonic functions as harmonic extensions of functions in the Sobolev space $H^{-1/2}(Z)$ by a Poisson operator \mathcal{P} . The main result says that, if A is a pseudo-differential operator of degree $d < 3$, the operator

$$B = \mathcal{P}^* \circ A \circ \mathcal{P}$$

is a pseudo-differential operator on Z of degree $d - 1$ whose principal symbol of degree $d - 1$ can be computed by integration of the principal symbol of A on the co-normal bundle of Z .

These “bilateral” extensions are simpler (at least for the Laplace operator) than the “unilateral” ones whose study is the theory of pseudo-differential operators on manifolds with boundary (see [1, 2, 3, 4, 6]).

1 Symbols

The following classes of symbols are defined in the books [4], sec. 7.1, and in [5], sec. 18.1. A *symbol of degree d* on $U_x \times \mathbb{R}_\xi^n$ where U is an open set in \mathbb{R}^N is a smooth complex valued function $a(x, \xi)$ on $U \times \mathbb{R}^n$ which satisfies the following estimates: for any multi-indices (α, β) , there exists a constant $C_{\alpha, \beta}$ so that

$$|D_x^\alpha D_\xi^\beta a(x; \xi)| \leq C_{\alpha, \beta} (1 + \|\xi\|)^{d - |\beta|}.$$

*Institut Mathématique de Jussieu, Université Pierre et Marie Curie

†Institut Fourier, Unité mixte de recherche CNRS-UJF 5582, BP 74, 38402-Saint Martin d'Hères Cedex (France); yves.colin-de-verdiere@ujf-grenoble.fr

The symbol a is called *classical* if a admits an expansion $a \sim \sum_{l=0}^{\infty} a_{d-l}$ where a_j is homogeneous of degree j (j an integer) for $\xi \in \mathbb{R}^n$ large enough; more precisely, for any $J \in \mathbb{N}$, $a - \sum_{j=0}^J a_{d-j}$ is a symbol of degree $d - J - 1$.

We will need the

Lemma 1 *If $a(x; \xi, \eta)$ is a symbol of degree $d < -1$ defined on $U_x \times (\mathbb{R}_\xi^n \times \mathbb{R}_\eta^n)$, $b(x; \xi) = \int_{\mathbb{R}} a(x; \xi, \eta) d\eta$ is a symbol of degree $d+1$ defined on $U_x \times \mathbb{R}_\xi^n$. Moreover, if a is classical, b is also classical and the homogeneous components of b are given for $l \leq d+1$, by $b_l(x; \xi) = \int_{\mathbb{R}} a_{l-1}(x; \xi, \eta) d\eta$*

2 A general reduction Theorem for pseudo-differential operators

We choose local coordinates in some neighborhood of a point in Z denoted $x = (z, y) \in \mathbb{R}^{d-1} \times \mathbb{R}$, so that $Z = \{y = 0\}$. We denote by $(\Omega_j, j = 1, \dots, N)$ a finite cover of Z by such charts and denote by Ω_0 an open set disjoint from Z so that $X = \cup_{j=0}^N \Omega_j$. We choose the charts Ω_j so that the densities $|dz|$ and $|dx|$ are the Lebesgue measures.

If X is a smooth manifold, we denote by $\mathcal{D}'(X)$ the space of generalized functions on X of which the space of smooth functions on X is a dense subspace. We assume that X and Z are equipped with smooth densities $|dx|$ and $|dz|$. This allows to identify generalized functions with Schwartz distributions, i.e. linear functionals on test functions; this duality extending the L^2 product is denoted by $\langle | \rangle$. We introduce the extension operator $\mathcal{E} : \mathcal{D}'(Z) \rightarrow \mathcal{D}'(X)$ sending the distribution f to the distribution $f\delta(y=0)$ defined

$$\langle f\delta(y=0) | \phi(z, y) \rangle = \langle f | \phi(z, 0) \rangle$$

and its adjoint, the trace $\mathcal{T} : C^\infty(X) \rightarrow C^\infty(Z)$ defined by $\phi \rightarrow \phi|_Z$. Let A be a pseudo-differential operator on X : let us call A_j the restriction of A to test functions compactly supported in Ω_j . We will work with one of the A_j 's given by the following “quantization” rule

$$A_j u(z, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{i(\langle z-z' | \zeta \rangle + (y-y')\eta)} a_j(z, y; \zeta, \eta) u(z', y') dz' dy' d\zeta d\eta .$$

So we have formally, using the facts that the densities on X and Z are given by the Lebesgue measures in these local coordinates:

$$\mathcal{T} \circ A_j \circ \mathcal{E} v(z) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d-1}} e^{i\langle z-z' | \zeta \rangle} a_j(z, 0; \zeta, \eta) v(z') dz' d\zeta d\eta ,$$

which we can rewrite

$$\mathcal{T} \circ A_j \circ \mathcal{E} v(z) = \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{2d}} e^{i\langle z-z' | \zeta \rangle} b_j(z; \zeta) v(z') dz' d\zeta ,$$

with

$$b_j(z; \zeta) = \frac{1}{2\pi} \int_{\mathbb{R}} a_j(z, 0; \zeta, \eta) d\eta . \quad (1)$$

We have the

Theorem 1 *If A is a pseudo-differential operator on X of degree $m < -1$ whose full symbol in the chart Ω_j is a_j , then the operator $\mathcal{T} \circ A \circ \mathcal{E}$ is a pseudo-differential operator on Z of degree $m + 1$ whose symbol is given in the charts $\Omega_j \cap Z$ by Equation (1).*

This is proved by looking at the actions on test functions compactly supported in the chart Ω_j , $j \geq 1$: then we use Lemma 1.

Remark 1 *The principal symbol can be described in a more intrinsic way: let $z \in Z$ be given, from the smooth densities on $T_z X$ and on $T_z Z$ given by $|dx|$ and $|dz|$, we get, using the Liouville densities, densities on the dual bundles $T_z^* Z$ and $T_z^* X$. Let us denote by $\Omega^1(E)$ the 1-dimensional space of densities on the vector space E . From the exact sequence*

$$0 \rightarrow N_z^* Z \rightarrow T_z^* X \rightarrow T_z^* Z \rightarrow 0 ,$$

we deduce

$$\Omega^1(T^* X) \equiv \Omega^1(N^* Z) \otimes \Omega^1(T^* Z)$$

and a canonical density $dm(z)$ in $\Omega^1(N_z^ Z)$. The principal symbol of $B = \mathcal{T} \circ A \circ \mathcal{E}$ is given in coordinates by $b(z, \zeta) = (1/2\pi) \int_{N_z^* Z} a(z; \zeta, \eta) dm(\eta)$.*

3 The “bilateral” Dirichlet-to-Neumann operator

We will assume that the local coordinates $x = (z, y)$ along Z are chosen so that $g(z, 0) = h(dz) + dy^2$ and the Riemannian volume along Z is $|dx|_g = |dz|_h |dy|$. We will choose the associated densities on X and Z . We will denote by Δ_g the Laplace-Beltrami operator on (X, g) as defined by Riemannian geometers (i.e. with a minus sign in front of the second order derivatives).

If f is given on Z , let us denote by $\mathcal{DN}(f)$ minus the sum of the interior normal derivatives on both sides of Z of the harmonic extension F of f ; this always makes sense, even if the normal bundle of Z is not orientable. We have the

Lemma 2 *The distributional Laplacian of the harmonic extension F of a smooth function f on Z is $\Delta_g F = \mathcal{E}(\mathcal{DN}(f))$.*

Proof. –

The proof is a simple application of the Green's formula: by definition of the action of the Laplacian on distributions, if ϕ is a test function on X , $\langle \Delta_g F | \phi \rangle := \langle F | \Delta_g \phi \rangle$. We can compute the righthandside integral as an integral on $X \setminus Z$ using Green's formula.

$$\int_{X \setminus Z} (F \Delta_g \phi - \phi \Delta_g F) |dx|_g = \int_Z (F \delta \phi - \phi \delta F) |dz|_h$$

where δ is the sum of the interior normal derivatives from both sides of Z . Using the fact that $\Delta_g F = 0$ in $X \setminus Z$ and $\delta \phi = 0$, we get the result. \square

Denoting by Δ_g^{-1} the “quasi-inverse” of Δ_g defined by $\Delta_g^{-1} \phi_j = \lambda_j^{-1} \phi_j$ for the eigenfunctions ϕ_j of Δ_g with non-zero eigenvalue λ_j and $\Delta_g^{-1} 1 = 0$, we have $f = (\mathcal{T} \circ \Delta_g^{-1} \circ \mathcal{E}) \circ \mathcal{DN}(f)$ (mod constants). By Theorem 1, the operator $B = \mathcal{T} \circ \Delta_g^{-1} \circ \mathcal{E}$ is an elliptic self-adjoint pseudo-differential operator on Z . The operator \mathcal{DN} is a right inverse of B modulo smoothing operators and hence also a left inverse modulo smoothing operators. So that $\mathcal{DN} = B^{-1}$ is an elliptic self-adjoint of principal symbol the inverse

$$\frac{1}{2\pi} \int_{\mathbb{R}} (\|\zeta\|_h^2 + \eta^2)^{-1} d\eta = \frac{1}{2\|\zeta\|_h} ,$$

namely $2\|\zeta\|_h$. Hence

Theorem 2 *The bilateral Dirichlet-to-Neumann \mathcal{DN} is a self-adjoint elliptic pseudo-differential operator of degree 1 on $L^2(Z, |dz|)$ and of principal symbol $2\|\zeta\|_h$. The kernel of \mathcal{DN} is the space of constant functions.*

The full symbol of \mathcal{DN} can be computed in a similar way from the full symbol of the resolvent Δ_g^{-1} along Z .

4 The Poisson operator

Let A be an pseudo-differential operator on X of principal symbol a . We are interested to the restriction to the space \mathcal{H} of the quadratic form $Q_A(F) = \langle AF | F \rangle$ associated to A . We will parametrize \mathcal{H} as harmonic extensions of functions which are in $H^{-\frac{1}{2}}(Z)$ by the so-called Poisson operator denoted by \mathcal{P} ; the pull-back R_A of Q_A on $L^2(Z)$ is defined by

$$R_A(f) = \langle A\mathcal{P}f | \mathcal{P}f \rangle = \langle \mathcal{P}^* A \mathcal{P} f | f \rangle .$$

The goal of this section is to compute the operator $B = \mathcal{P}^* A \mathcal{P}$ associated to the quadratic form R_A .

From Lemma 2, we have, modulo smoothing operators,

$$\mathcal{P} = \Delta_g^{-1} \circ \mathcal{E} \circ \mathcal{DN} .$$

Hence

$$B = \mathcal{DN} \circ [\mathcal{T} \circ (\Delta_g^{-1} \circ A \circ \Delta_g^{-1}) \circ \mathcal{E}] \circ \mathcal{DN} .$$

The operator $\Delta_g^{-1} \circ A \circ \Delta_g^{-1}$ is a pseudo-differential operator of principal symbol $a/(\|\zeta\|_h^2 + \eta^2)^2$ near Z .

Applying Theorem 1 to the inner bracket and Theorem 2, we get the:

Theorem 3 *If A is a pseudo-differential operator of degree $d < 3$ on X and \mathcal{P} the Poisson operator associated to Z , the operator $B = \mathcal{P}^* A \mathcal{P}$ is a pseudo-differential operator of degree $d - 1$ on Z of principal symbol*

$$b(z, \zeta) = \frac{2}{\pi} \|\zeta\|_h^2 \int_{\mathbb{R}} \frac{a(z, 0; \zeta, \eta)}{(\|\zeta\|_h^2 + \eta^2)^2} d\eta .$$

Remark 2 *Note that if A is a pseudo-differential operator without the transmission property, the operator $A \circ \mathcal{P}$ may be ill-behaved and have disagreeable singularities along Z ; however $\mathcal{P}^* A \mathcal{P}$ is always a good pseudo-differential operator on Z .*

References

- [1] Alberto Calderón. Boundary value problems for elliptic equations. *Outlines of the joint Soviet-American Symposium on Partial Differential Equations, Novosibirsk*, pp 303–304 (1963).
- [2] Louis Boutet de Monvel. Comportement d'un opérateur pseudo-différentiel sur une variété à bord I, II. *J. Anal. Math.* **17**:241–304 (1966).
- [3] Louis Boutet de Monvel. Boundary problems for pseudo-differential operators. *Acta Mathematica* **126**:11–51 (1971).
- [4] Gerd Grubb. Distributions and Operators. *Springer* (2008).
- [5] Lars Hörmander. The Analysis of Linear Partial Differential Operators III. Pseudo-Differential Operators. *Grundlehren der Mathematischen Wissenschaften (Springer)* **274** (1985).
- [6] Michael E. Taylor. Partial Differential Equations II, Qualitative study of Linear Equations. *Springer, Applied Math. Sciences* **116** (1996).